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Theory of spontaneous radiation by electrons in a trajectory-coherent approximation

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Abstract. The first-order quantum correction for the characterization of spontaneous radiation is calculated by means of electron quasi-classical trajectory-coherent states in an arbitrary electromagnetic field. Well known expressions for the characterization of spontaneous radiation are obtained using quasi-classical approximation. The first-order quantum correction is derived as a functional from a classical trajectory (among which is a classical spin vector). Transitions with spin flip and without spin flip are distinguished. Those elements connected with photon kick and quantum motion characteristics are selected for first-order quantum correction. It is shown that, using an ultra-relativistic approximation, the latter may be ignored, but when using a non-relativistic approximation their contributions are approximately equal. A special trajectory-coherent representation that significantly simplifies the investigation of spontaneous radiation is proposed.

Introduction

Quantum electrodynamics allows one, in principle, to obtain the general solution of the problem describing spontaneous radiation of electromagnetic waves by charged particles moving in external fields [1]. However, theoretical results in a simple and visual form can be obtained comparatively rarely [2–25]. Thus the development of effective approximate methods of theoretical analysis for the problem of spontaneous electromagnetic radiation is still an urgent problem. One of the most developed theories is the theory of synchrotron radiation [2–6], for radiation in undulators [7–10], for radiation under axial channelling [9–12] and in systems with a quadratic Hamiltonian [13–15].

The analysis carried out in [4, 5] showed that when ultra-relativistic particles are considered one can, as a rule, neglect the 'quantum character' of the particle trajectory and consider only the photon loss§. In particular, it was shown how the classical formulas for the characteristics of spontaneous radiation of a pointwise charge could be obtained, including an exact classical expression for the Fourier transform of the Lie'nard–Wiechert potentials (see, for example, [5, p 144]). In this case the assumption that there exist quantum-mechanical states concentrated near the classical trajectory of a particle is essential. However, the explicit form of such states was not represented in

§ This theoretical analysis was made more precise in [26, 27]. There the natural questions concerning the initial conditions arise in the method accounting for the loss of the radiated photon.

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[5]. A complete and orthonormalized set of such states, which are approximate (quasi-classical) solutions of the Klein–Gordon and Schrödinger equations, was constructed in [28–30], and for the Dirac equation such a set was constructed in [31–34]; these states were called *trajectory-coherent* (TCS).

By using these states, in [35] for a magnetic undulator and in [36] in the general case, the procedure for obtaining exact expressions for the Fourier transforms of the Lie'nard–Wiechert potentials (in the wavezone) from quantum theory has been performed. In [35, 36] it was also shown (by using specific examples) how one can write the first quantum correction to the radiation power (or the energy radiated) of the spinless particle in the form of a specific functional of a classical trajectory of a particle.

Here we show that the quasi-classical trajectory-coherent states [31, 33] for the Dirac equation in an arbitrary external field allow one to obtain the first quantum correction to the radiation of a charged spinor particle in the form of a certain specific functional of a classical trajectory. However, as a classical trajectory, we now consider not only the solution of the classical Lorentz equations, but also the solution of the classical spin equations.

1. Quasi-classical trajectory-coherent states of an electron in an arbitrary electromagnetic field

The construction of quasi-classical trajectory-coherent states (TCS) of an electron in an arbitrary external field, which are asymptotic solutions of the Dirac equation with accuracy to any power of $\hbar$ as $\hbar \to 0$, was presented in detail in [31, 33]. It turns out that for the purposes of our work, i.e. for obtaining the guaranteed first quantum corrections to the characteristics of spontaneous radiation, it is sufficient to use the electron TCS constructed to an accuracy $O(\hbar^{5/2})$. We present the explicit form of the corresponding trajectory-coherent states of electron, following [31].

The motion of a relativistic charged particle will be described by the Dirac equation

$$[-i\hbar\partial_t + \hat{\mathscr{H}}]\Psi = 0 \tag{1.1}$$

where the Hamiltonian has the form

$$\hat{\mathscr{H}} = c\alpha\hat{\mathscr{P}} + \rho_3 m_0 c^2 + e\Phi(x, t) \tag{1.2}$$

where $\hat{\mathscr{P}} = -i\hbar\nabla - (e/c)A(x, t)$, and $A_\mu = (\Phi, -A)$ is the electromagnetic potential (arbitrary smooth functions in $x \in R^3$, $t \in R^1$) and they increase together with their derivatives as $|x| \to \infty$ not greater than a certain power of $|x|$ uniformly in $t \in R^1$. For Hermitian Dirac matrices $\alpha = \rho_1\Sigma$, Σ, $\rho_k (k = 1, 2, 3)$ we use the standard representation, and ac is a Euclidian scalar product

$$ac = \langle ac \rangle = \sum_{i=1}^{3} a_i c_i .$$

The main symbol of the Hamiltonian operator $\hat{\mathscr{H}}$ (1.2) has the form $\mathscr{H}(p, x, t) = c\alpha\hat{\mathscr{P}} + \rho_3 m_0 c^2 + e\Phi(x, t)$, where $\mathscr{P} = p - (e/c)A(x, t)$.

The quasi-classical positive frequency TCS satisfying the Dirac equation (1.1) with accuracy to $O(\hbar^{5/2})$ has the form

$$\Psi_{\nu,\zeta}(x, t, \hbar) = \hat{\mathscr{K}}_t^{(2)}(\hbar) | H_\nu, \zeta \rangle \tag{1.3}$$

where

$$\hat{\mathscr{K}}_t^{(2)}(\hbar)=\hat{\mathscr{K}}_t^{(0)}(\hbar)\left\{\Pi_+(t)\left[1+\frac{\hbar}{2}\left(\frac{1}{2\varepsilon(t)}\hat{Q}_1\right)^2\right]\right.$$
$$\left.+\frac{\sqrt{\hbar}}{2\varepsilon(t)}\Pi_-(t)[\hat{Q}_1+\sqrt{\hbar}\hat{Q}_2+\hbar\hat{Q}_3+\hbar^{3/2}\hat{Q}_4]\right\}[1-\mathrm{i}\sqrt{\hbar}\hat{\pi}_1-\mathrm{i}\hbar\hat{\pi}_2-\hbar\hat{\pi}_1^2]. \quad (1.4)$$

$$\mathscr{H}(p,x,t)\Pi_\pm(p,x,t)=\lambda^\pm(p,x,t)\Pi_\pm(p,x,t)$$
$$\Pi_+(p,x,t)=\frac{1}{\sqrt{2\varepsilon(\varepsilon+m_0c^2)}}\begin{pmatrix}\varepsilon+m_0c^2\\ c\sigma\mathscr{P}\end{pmatrix}$$
$$\Pi_-(p,x,t)=\frac{1}{\sqrt{2\varepsilon(\varepsilon+m_0c^2)}}\begin{pmatrix}c\sigma\mathscr{P}\\ -\varepsilon-m_0c^2\end{pmatrix} \quad (1.5)$$
$$\lambda^\pm(p,x,t)=e\Phi(x,t)\pm\varepsilon(p,x,t)$$
$$\varepsilon(p,x,t)=(c^2(\mathscr{P},\mathscr{P})+m_0^2c^4)^{1/2}.$$

The matrices $\Pi_\pm(t)=\Pi_\pm(p(t),x(t),t)$ are calculated† at the point $r_t(x_0,p_0)=(x(t,x_0,p_0),p(t,x_0,p_0))$, where the functions $x(t,x_0,p_0),p(\mathrm{t},x_0,p_0)$ are solutions of the classical Hamiltonian system‡

$$\dot{p}=-\lambda_x(p,x,t) \qquad p(0)=p_0$$
$$\dot{x}=\lambda_p(p,x,t) \qquad x(0)=x_0. \quad (1.6)$$

The operator $\hat{\mathscr{K}}_t^{(0)}(\hbar)$ is equal to

$$\hat{\mathscr{K}}_t^{(0)}(\hbar)(\cdot)=N_0(\hbar)[\det C(t)]^{-1/2}\exp\left\{\frac{\mathrm{i}}{\hbar}S(x,t)\right\}(\cdot). \quad (1.7)$$

Here

$$N_0(\hbar)=\left(\prod_{j=1}^{3}\operatorname{Im} b_j(\pi\hbar)^{-3}\right)^{1/4}$$

is the normalization constant, the phase $S(x,t)$ is a 'complex' action [37] of the form

$$S(x,t)=\int_0^t[\langle\dot{x}(t),p(t)\rangle-\lambda(t)]\,\mathrm{d}t+\langle p(t),\Delta x\rangle+\tfrac{1}{2}\langle\Delta x,B(t)\cdot C^{-1}(t)\Delta x\rangle$$

where $\Delta x=x-x(t)$, and 3×3-complex matrices

$$B(t)=[w_1(t),w_2(t),w_3(t)] \qquad C(t)=[z_1(t),z_2(t),z_3(t)]$$

are solutions of the system in variations (this is the linearization of the Hamiltonian system (1.6) in the neighbourhood of the trajectory $r_t(x_0,p_0)$, $t\in R^1$)

† The dependence of values calculated at the points of a classical trajectory on x_0 and p_0 can be omitted below

$$\varepsilon(t)=\varepsilon(x(t,x_0,p_0),p(t,x_0,p_0),t).$$

‡ Here and below, the index (+) in the classical Hamiltonian function will be omitted

$$\lambda(p,x,t)=\lambda^{(+)}(p,x,t).$$

$$\dot{B}=-\lambda_{xp}(t)B-\lambda_{xx}(t)C$$
$$\dot{C}=\lambda_{pp}(t)B+\lambda_{px}(t)C \tag{1.8}$$
$$C(0)=\|\delta_{ij}\| \qquad B(0)=\|b_j\delta_{ij}\| \qquad \mathrm{Im}\, b_j>0\ (i,j=1,2,3)$$

The function $|H_\nu, \zeta\rangle$ has the form

$$|H_\nu, \zeta\rangle = H_\nu \cdot u(t,\zeta) \qquad \nu=(\nu_1, \nu_2, \nu_3)$$
$$H_\nu=\prod_{j=1}^{3}(\nu_j!)^{-1/2}(\hat{\Lambda}_j^+)^{\nu_j}\cdot 1 \qquad \nu_j=0,1,2,3,\ldots \tag{1.9}$$

where $(\hat{\Lambda}_1^+, \hat{\Lambda}_2^+, \hat{\Lambda}_3^+)=\hat{\Lambda}^+$ and $(\hat{\Lambda}_1, \hat{\Lambda}_2, \hat{\Lambda}_3)=\hat{\Lambda}$ are the 'annihilation' and 'creation' operators [38]

$$\begin{pmatrix}\hat{\Lambda}\\ \hat{\Lambda}^+\end{pmatrix}=\hat{T}\begin{pmatrix}\widehat{\Delta p}\\ \Delta x\end{pmatrix} \qquad \hat{T}=\frac{1}{\sqrt{2\hbar}}\begin{pmatrix}\sqrt{D_0}C^t & -\sqrt{D_0}B^t\\ \sqrt{D_0}C^+ & -\sqrt{D_0}B^+\end{pmatrix}$$
$$\widehat{\Delta p}=-\mathrm{i}\hbar\nabla - B(t)C^{-1}(t)\Delta x \qquad D_0=\left\|\frac{\delta_{ij}}{\mathrm{Im}\, b_j}\right\|_{3\times 3} \tag{1.10}$$

and the spinor $u(t,\zeta)$ satisfies the following equation

$$\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{ec}{2\varepsilon(t)}\left(\langle\boldsymbol{\sigma}, \boldsymbol{H}(t)\rangle+\frac{1}{1+\gamma^{-1}}\langle\boldsymbol{\sigma}, \boldsymbol{\beta}\times\boldsymbol{E}(t)\rangle\right)\right]u=0$$
$$c\boldsymbol{\beta}=\dot{\boldsymbol{x}}(t) \qquad \gamma^{-1}=\sqrt{1-\boldsymbol{\beta}^2}$$

with initial conditions [39]

$$\langle\boldsymbol{\sigma}, \boldsymbol{l}\rangle u(0,\zeta)=\zeta u(0,\zeta) \qquad \zeta=\pm 1 \tag{1.11}$$

which fixes the particle spin direction along the unit vector $\boldsymbol{l}\in R^3$ for $t=0$; here $\boldsymbol{\sigma}=(\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. Then the operators $\hat{Q}_j$ and $\hat{\pi}_j$ can be represented in the form†

$$\sqrt{\hbar}\hat{Q}_1=c\left(\langle\boldsymbol{\sigma},\boldsymbol{\beta}\rangle\frac{\langle\boldsymbol{\beta},\hat{\mathscr{P}}_1\rangle}{1+\gamma^{-1}}-\langle\boldsymbol{\sigma},\hat{\mathscr{P}}_1\rangle\right) \qquad \hat{\mathscr{P}}_1=\widehat{\Delta p}-\frac{e}{c}d^1\mathscr{A}$$

$$\hbar\hat{Q}_2=-\frac{e}{2}\left(\langle\boldsymbol{\sigma},\boldsymbol{\beta}\rangle\frac{\langle\boldsymbol{\beta},d^2\mathscr{A}\rangle}{1+\gamma^{-1}}-\langle\boldsymbol{\sigma},d^2\mathscr{A}\rangle\right)$$
$$+\frac{\mathrm{i}\hbar}{2}\left(\langle\boldsymbol{\sigma},\dot{\boldsymbol{\beta}}\rangle+\langle\boldsymbol{\sigma},\boldsymbol{\beta}\rangle\frac{\gamma\langle\boldsymbol{\beta},\dot{\boldsymbol{\beta}}\rangle}{1+\gamma^{-1}}\right)-\sqrt{\hbar}\frac{c}{\varepsilon}\langle\boldsymbol{\beta},\hat{\mathscr{P}}_1\rangle\hat{Q}_1$$

$$\hat{\pi}_j\varphi=\sum_{|\nu|=0}^{\infty}\sum_{\zeta=\pm1}|H_\nu,\zeta\rangle\int_0^t \mathrm{d}\tau\,\langle\zeta, H_\nu|\hat{F}_j|\varphi\rangle \qquad j=1,2$$

$$\hbar^{3/2}\hat{F}_1=\frac{e}{3!}(d^3\Phi-\langle\boldsymbol{\beta},d^3\mathscr{A}\rangle)+\frac{\hbar}{2\varepsilon}(\hat{Q}_1\cdot\hat{Q}_2+\hat{Q}_2^+\hat{Q}_1)+\frac{\hbar}{2\varepsilon^2}\hat{Q}_1\cdot\langle\boldsymbol{\beta},\hat{\mathscr{P}}_1\rangle\hat{Q}_1$$

† The operators $\hat{Q}_3$, $\hat{Q}_4$, and $\hat{F}_2$ do not affect the first quantum correction to the power of spontaneous radiation, therefore we do not present them here explicitly (see [31]).

$$d^k\mathscr{A}=d^k\mathscr{A}(t)=\left(\left\langle\Delta x,\frac{\partial}{\partial y}\right\rangle\right)^k\mathscr{A}(y,t)\Big|_{y=x(t)}.$$

Here $\langle\varphi_1\,|\,\varphi_2\rangle$ denotes the scalar product in the Hilbert space $L^t_\hbar$ [32]

$$\langle\varphi_1\,|\,\varphi_2\rangle=\int_{R^3}\mathrm{d}^3x\,\rho_\hbar(x,t)\varphi_1^+\cdot\varphi_2\qquad \rho_\hbar(x,t)=(\hat{\mathscr{K}}_t^{(0)}(\hbar))^+\hat{\mathscr{K}}_t^{(0)}(\hbar). \tag{1.12}$$

Note that the operators Δx_j, $\widehat{\Delta p}_j$, $\hat{Q}_1$, $\hat{F}_1$, and $\hat{F}_2$ are self-adjoint in $L^t_\hbar$, and the functions (1.9) form in $L^t_\hbar$ a complete orthonormalized set of states

$$\langle\zeta',H_{\nu'}\,|\,H_\nu,\zeta\rangle=\delta_{\nu,\nu'}\delta_{\zeta,\zeta'}. \tag{1.13}$$

There are no difficulties in calculating the matrix element in an arbitrary operator $\widehat{\mathscr{A}_\hbar}$ with precision $O(\hbar^{(N+1)/2})$ with respect to the quasi-classical TC-states. For this purpose it is necessary to obtain the operator in the quasi-classical TC-representation up to $\hat{O}(\hbar^{(N+1)/2})$ [31] and average the expression obtained with respect to the wavefunction $|H_\nu,\zeta\rangle$ (function 1.9) taking into account (1.13) and

$$\begin{aligned}\hat{\Lambda}_j^+H_{\nu_1,\nu_2,\nu_3}&=(\nu_j+1)^{1/2}H_{\nu_1+\delta_{1j},\nu_2+\delta_{2j},\nu_3+\delta_{3j}}\\ \hat{\Lambda}_jH_{\nu_1,\nu_2,\nu_3}&=\sqrt{\nu_j}H_{\nu_1-\delta_{1j},\delta_2-\delta_{2j},\nu_3-\delta_{3j}}\end{aligned} \tag{1.14}$$

$$u^+(t,\zeta')\sigma u(t,\zeta)=\boldsymbol{\eta}(t,\zeta,\zeta'). \tag{1.15}$$

Here $\boldsymbol{\eta}(t,\zeta,\zeta')$ is the solution of the Bargmann–Michel–Telegdi equation for $g=2$ [40]

$$\dot{\boldsymbol{\eta}}=\frac{ec}{\varepsilon}\boldsymbol{\eta}\times\left[\boldsymbol{H}(t)+\frac{1}{1+\gamma^{-1}}\boldsymbol{\beta}\times\boldsymbol{E}(t)\right] \tag{1.16}$$

with the initial condition

$$\boldsymbol{\eta}(0,\zeta,\zeta')=\frac{1+\zeta\zeta'}{2}\zeta\boldsymbol{l}+\frac{1-\zeta\zeta'}{2}\frac{\boldsymbol{l}\times(\boldsymbol{k}\times\boldsymbol{l})+\mathrm{i}\zeta\boldsymbol{l}\times\boldsymbol{k}}{(1-(\boldsymbol{l},\boldsymbol{k})^2)^{1/2}},$$

where $\boldsymbol{k}=(0,0,1)$, and $\boldsymbol{l}$ was defined in (1.11).

In order to use (1.13)–(1.15), we present the relations expressing the operators Δx_j and $\widehat{\Delta p}_j$ in terms of the operators of 'creation' $\hat{\Lambda}_j^+$ and 'annihilation' $\hat{\Lambda}_j$ (1.10) [41]

$$\begin{pmatrix}\widehat{\Delta p}\\ \Delta x\end{pmatrix}=\hat{T}^{-1}\begin{pmatrix}\hat{\Lambda}\\ \hat{\Lambda}^+\end{pmatrix}\qquad \hat{T}^{-1}=\mathrm{i}\sqrt{\frac{\hbar}{2}}\begin{pmatrix}B^*\sqrt{D_0} & -B\sqrt{D_0}\\ C^*\sqrt{D_0} & -C\sqrt{D_0}\end{pmatrix}. \tag{1.17}$$

Since the inverse matrix $T\cdot T^{-1}=T^{-1}\cdot T=T_{6\times 6}$, by (1.10) and (1.17) it is easy to obtain the following matrix relations which will be used below

$$\begin{aligned}&B^*D_0C^t-BD_0C^+=-2\mathrm{i}I_{3\times 3}\\ &C^*D_0C^t-CD_0C^+=B^*D_0B^t-BD_0B^+=0.\end{aligned} \tag{1.18}$$

Relations (1.13)–(1.15), (1.17) and the explicit form of the operator $\hat{\mathscr{K}}_t^{(N)}(\hbar)$ defining the transition to the quasi-classical TC-representation up to mod $O(\hbar^{(N+1)/2})$ allow us to calculate, in principle, the matrix elements of an arbitrary operator $\hat{\mathscr{A}}_\hbar(t)$ with precision $O(\hbar^{(N+1)/2})$ if its symbol $\mathscr{A}(\boldsymbol{p},\boldsymbol{x},t)$ is a smooth function in $\boldsymbol{p}$, $\boldsymbol{x}$ and t, together with all its derivatives.

2. Transition current operator in quasi-classical trajectory-coherent representation

The operator $\hat{\mathscr{K}}_t^{(2)}(\hbar)$ (1.4) defines mod $O(\hbar^{3/2})$ the transition to the quasi-classical TC-representation

$$\varphi=(\hat{\mathscr{K}}_t^{(2)}(\hbar))^{-1}\Psi+O(\hbar^{3/2})$$

$$\hat{\mathscr{A}}_+=(\hat{\mathscr{K}}_t^{(2)}(\hbar))^{-1}\hat{\mathscr{A}}_\hbar\hat{\mathscr{K}}_t^{(2)}(\hbar)+\hat{O}(\hbar^{3/2}).$$

We calculate the current operator in this representation

$$\hat{j}_+=\boldsymbol{\alpha}\exp\left\{\mathrm{i}\omega\left(t-\frac{1}{c}\langle\boldsymbol{n},\boldsymbol{x}\rangle\right)\right\}$$

$$\boldsymbol{n}=(n_1,n_2,n_3)=(\cos\varphi\sin\theta,\sin\varphi\sin\theta,\cos\theta).$$

By (1.4), we obtain

$$\begin{aligned}\hat{j}_+&=(\hat{\mathscr{K}}_t^{(2)}(\hbar))^{-1}\boldsymbol{\alpha}\exp\left\{\mathrm{i}\omega\left(t-\frac{1}{c}\langle\boldsymbol{n},\boldsymbol{x}\rangle\right)\right\}\hat{\mathscr{K}}_t^{(2)}(\hbar)+\hat{O}(\hbar^{3/2})\\
&=(1+\mathrm{i}\sqrt{\hbar}\hat{\pi}_1^++\mathrm{i}\hbar\hat{\pi}_2^+-\hbar(\hat{\pi}_1^+)^2)\left\{\left[1-\frac{\hbar}{2}\left(\frac{1}{2\varepsilon}\hat{Q}_1^+\right)^2\right]\boldsymbol{\beta}\exp\left\{\mathrm{i}\omega\left(t-\frac{1}{c}\langle\boldsymbol{n},\boldsymbol{x}\rangle\right)\right\}\right.\\
&\quad\times\left[1-\frac{\hbar}{2}\left(\frac{1}{2\varepsilon}\hat{Q}_1\right)^2\right]-\frac{\sqrt{\hbar}}{2\varepsilon}\hat{Q}_1^+\boldsymbol{\beta}\exp\left\{\mathrm{i}\omega\left(t-\frac{1}{c}\langle\boldsymbol{n},\boldsymbol{x}\rangle\right)\right\}\frac{\sqrt{\hbar}}{2\varepsilon}\hat{Q}_1\\
&\quad+\left(\frac{\boldsymbol{\beta}}{1+\gamma^{-1}}\langle\boldsymbol{\sigma},\boldsymbol{\beta}\rangle-\boldsymbol{\sigma}\right)\exp\left\{\mathrm{i}\omega\left(t-\frac{1}{c}\langle\boldsymbol{n},\boldsymbol{x}\rangle\right)\right\}\frac{\sqrt{\hbar}}{2\varepsilon}(\hat{Q}_1+\sqrt{\hbar}\hat{Q}_2)\\
&\quad\left.+\frac{\sqrt{\hbar}}{2\varepsilon}(\hat{Q}_1^++\sqrt{\hbar}\hat{Q}_2^+)\left(\frac{\boldsymbol{\beta}}{1+\gamma^{-1}}\langle\boldsymbol{\sigma},\boldsymbol{\beta}\rangle-\boldsymbol{\sigma}\right)\exp\left\{\mathrm{i}\omega\left(t-\frac{1}{c}\langle\boldsymbol{n},\boldsymbol{x}\rangle\right)\right\}\right\}\\
&\quad\times(1+\mathrm{i}\sqrt{\hbar}\hat{\pi}_1-\hbar(\mathrm{i}\hat{\pi}_2+\hat{\pi}_1^2))+\hat{O}(\hbar^{3/2}).\end{aligned}\tag{2.1}$$

We denote by $\hat{O}(\hbar^\alpha)$ the operator $\hat{F}: L_\hbar^t\to L_\hbar^t$ for which $\|\widehat{F\varphi}\|_{L_\hbar^t}=O(\hbar^\alpha)$, as $\hbar\to0$, $\varphi\in L_\hbar^t$. Since the measure (1.12) depends on a small parameter, in all subsequent calculations one can approximate the smooth functions $\varphi(\boldsymbol{x},t)$ by the partial sums of the Taylor series in powers of $\Delta\boldsymbol{x}=\hat{O}(\sqrt{\hbar})$ with the given accuracy in $\hbar\to0$.

By using this fact, we can obtain the following expressions for the coordinate and velocity operators in TC-representation

$$\begin{aligned}\hat{\boldsymbol{X}}_+&=(\hat{\mathscr{K}}_t^{(2)}(\hbar))^{-1}\boldsymbol{x}\hat{\mathscr{K}}_t^{(2)}(\hbar)\\
&=\boldsymbol{x}(t)+\Delta\boldsymbol{x}-\mathrm{i}\sqrt{\hbar}(\Delta\boldsymbol{x}\hat{\pi}_1-\hat{\pi}_1^+\Delta\boldsymbol{x})+\hat{O}(\hbar^{3/2})\end{aligned}\tag{2.2}$$

$$\hat{\dot{X}}_{+}=(\hat{\mathscr{K}}_{t}^{(2)}(\hbar))^{-1}c\alpha\hat{\mathscr{K}}_{t}^{(2)}(\hbar)=\dot{x}\left(1-\frac{1}{2\varepsilon}\langle\hat{\mathscr{P}}_{1},\lambda_{pp}\hat{\mathscr{P}}_{1}\rangle\right)$$
$$+\lambda_{pp}\hat{\mathscr{P}}_{1}-\frac{e}{c}d^{2}\mathscr{A}-\frac{1}{2\varepsilon}(\langle\dot{x},\hat{\mathscr{P}}_{1}\rangle\lambda_{pp}\hat{\mathscr{P}}_{1}+\lambda_{pp}\hat{\mathscr{P}}_{1}\langle\dot{x},\hat{\mathscr{P}}_{1}\rangle)$$
$$-\mathrm{i}\sqrt{\hbar}[\lambda_{pp}\hat{\mathscr{P}}_{1}\hat{\pi}_{1}-\hat{\pi}_{1}^{+}\lambda_{pp}\hat{\mathscr{P}}_{1}]+\frac{ec\hbar}{2\varepsilon^{2}}\left[\dot{x}\left(\gamma^{-1}\langle\boldsymbol{\sigma},\boldsymbol{H}\rangle+\langle\boldsymbol{\sigma},\boldsymbol{\beta}\rangle\frac{\langle\boldsymbol{\beta},\boldsymbol{H}\rangle}{1+\gamma^{-1}}\right)\right.$$
$$\left.-c\left(\frac{\boldsymbol{\beta}}{1+\gamma^{-1}}\langle\boldsymbol{\sigma},\boldsymbol{\beta}\times\boldsymbol{E}\rangle+\boldsymbol{\sigma}\times\boldsymbol{E}+\boldsymbol{\sigma}\times\boldsymbol{\beta}\frac{\gamma\langle\boldsymbol{\beta},\boldsymbol{E}\rangle}{1+\gamma^{-1}}\right)\right]+\hat{O}(\hbar^{3/2}). \qquad (2.3)$$

By substituting these expressions into (2.1) for the transition current operator in the TC-representation (mod $\hat{O}(\hbar^{3/2})$), we obtain

$$\hat{\boldsymbol{j}}_{+}=\frac{1}{2c}\left[\hat{\dot{X}}_{+}\exp\left\{\mathrm{i}\omega\left(t-\frac{1}{c}\langle\boldsymbol{n},\hat{X}_{+}\rangle\right)\right\}+\exp\left\{\mathrm{i}\omega\left(t-\frac{1}{c}\langle\boldsymbol{n},\hat{X}_{+}\rangle\right)\right\}\hat{\dot{X}}_{+}\right]$$
$$+\frac{\mathrm{i}\omega\hbar}{2\varepsilon}\left[\boldsymbol{\sigma}\times\boldsymbol{n}-\frac{\boldsymbol{\beta}}{1+\gamma^{-1}}\langle\boldsymbol{\sigma},\boldsymbol{n}\times\boldsymbol{\beta}\rangle-\boldsymbol{\sigma}\times\boldsymbol{\beta}\frac{\langle\boldsymbol{\beta},\boldsymbol{n}\rangle}{1+\gamma^{-1}}\right]$$
$$\times\exp\left\{\mathrm{i}\omega\left(t-\frac{1}{c}\langle\boldsymbol{n},\boldsymbol{x}(t)\rangle\right)\right\}+\hat{O}(\hbar^{3/2}). \qquad (2.4)$$

3. Spectral-angle distribution of energy of spontaneous radiation

We shall find the matrix elements of the transition current $\hat{\boldsymbol{j}}_{+}$ (2.4) and (1.18); we have

$$M(t,\nu,\zeta,\nu',\zeta')$$
$$=\langle\zeta',H_{\nu'}|\hat{\boldsymbol{j}}_{+}|H_{\nu},\zeta\rangle$$
$$=\exp\left\{\mathrm{i}\omega t-\frac{\mathrm{i}\omega}{c}\langle\boldsymbol{n},\boldsymbol{x}(t,\zeta,\zeta',\hbar)\rangle\right\}\left\{\frac{1}{c}\dot{\boldsymbol{x}}(t,\zeta,\zeta',\hbar)\delta_{\nu,\nu'}\right.$$
$$-\mathrm{i}\sqrt{\frac{\hbar}{2c^{2}}}[\dot{C}(t)\boldsymbol{\mu}_{+}+\mathrm{i}\omega\boldsymbol{\beta}\langle\boldsymbol{n},C(t)\boldsymbol{\mu}_{+}\rangle]\delta_{\zeta,\zeta'}$$
$$+\mathrm{i}\sqrt{\frac{\hbar}{2c^{2}}}[\dot{C}^{*}(t)\boldsymbol{\mu}_{-}+\mathrm{i}\omega\boldsymbol{\beta}\langle\boldsymbol{n},C^{*}(t)\boldsymbol{\mu}_{-}\rangle]\delta_{\zeta,\zeta'}$$
$$-\delta_{\nu,\nu'}\delta_{\zeta,\zeta'}\frac{1}{c^{2}}\left[\frac{\omega^{2}}{2}\boldsymbol{\beta}\langle\boldsymbol{n},\sigma_{xx}\boldsymbol{n}\rangle+\mathrm{i}\omega\lambda_{pp}\sigma_{px}\boldsymbol{n}+\mathrm{i}\omega\lambda_{px}\sigma_{xx}\boldsymbol{n}\right]$$
$$+\frac{\mathrm{i}\omega\hbar}{2\varepsilon}\left[\boldsymbol{\eta}\times\boldsymbol{n}-\frac{\boldsymbol{\beta}}{1+\gamma^{-1}}\langle\boldsymbol{\eta},\boldsymbol{n}\times\boldsymbol{\beta}\rangle-\boldsymbol{\eta}\times\boldsymbol{\beta}\frac{\langle\boldsymbol{n},\boldsymbol{\beta}\rangle}{1+\gamma^{-1}}\right]\delta_{\nu,\nu'}$$
$$\left.+\hbar\mathscr{L}(t,\nu,\zeta,\zeta')\delta_{|\nu|,|\nu'|+2}\right\}+O(\hbar^{3/2}). \qquad (3.1)$$

Here

$$\mu_{\pm}=\left\{\left(\sqrt{\frac{2\nu_1+1\pm1}{2\,\mathrm{Im}\,b_1}}\,\delta_{\nu_1',\nu_1\pm1}\delta_{\nu_2',\nu_2}\delta_{\nu_3',\nu_3}\right)\qquad\left(\sqrt{\frac{2\nu_2+1\pm1}{2\,\mathrm{Im}\,b_2}}\,\delta_{\nu_1',\nu_1}\delta_{\nu_2',\nu_2\pm1}\delta_{\nu_3',\nu_3}\right)\right.$$
$$\left.\left(\sqrt{\frac{2\nu_3+1\pm1}{2\,\mathrm{Im}\,b_3}}\,\delta_{\nu_1',\nu_1}\delta_{\nu_2',\nu_2}\delta_{\nu_3',\nu_3\pm1}\right)\right\}.$$

We do not present the explicit form of $\mathscr{L}(t,\nu,\zeta,\zeta')$, since one can show that the quantum correction to the matrix element proportional to the Kronecker symbol $\delta_{|\nu|,|\nu'|+2}$ does not effect the character of the radiation within the required accuracy in $\hbar\to0$. The mean values of the operators X_+ (2.2) and $\dot{X}_+$ (2.3) in (3.1) with respect to functions (1.9) have the form

$$\begin{aligned}\langle\hat{X}_+\rangle&=\boldsymbol{x}(t,\zeta,\zeta',\hbar)=\langle\zeta',H_\nu|\hat{X}_+|H_\nu,\zeta\rangle\\&=\boldsymbol{x}(t)\delta_{\zeta,\zeta'}-\mathrm{i}\sqrt{\hbar}\langle\zeta',H_\nu|(\Delta\boldsymbol{x}\hat{\pi}_1-\hat{\pi}_1^{+}\Delta\boldsymbol{x})|H_\nu,\zeta\rangle+O(\hbar^2)\end{aligned}\tag{3.2}$$

$$\begin{aligned}\langle\hat{\dot{X}}_+\rangle&=\dot{\boldsymbol{x}}(t,\zeta,\zeta',\hbar)=\langle\zeta',H_\nu|\hat{\dot{X}}_+|H_\nu,\zeta\rangle=\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t,\zeta,\zeta',\hbar)\\&=\left\{c\boldsymbol{\beta}-\sum_{k=1}^{3}\left(\frac{\hbar}{4}c\boldsymbol{\beta}\frac{2\nu_k+1}{c^2\,\mathrm{Im}\,b_k}(|\dot{z}_k|^2+\gamma^2|\boldsymbol{\beta}\dot{z}_k|^2)\right.\right.\\&\quad-\frac{\hbar}{2}\frac{2\nu_k+1}{c\,\mathrm{Im}\,b_k}\gamma^2\,\mathrm{Re}|\dot{z}_k\langle\boldsymbol{\beta}\dot{z}_k^*\rangle|^2)\\&\quad\left.\left.+\frac{e}{3c}\frac{\hbar(2\nu_k+1)}{2\,\mathrm{Im}\,b_k}\mathrm{Re}\{\langle z_k\nabla\rangle\lambda_{pp}(t)z_k^*\times\boldsymbol{H}(\boldsymbol{x},t)\}\bigg|_{\boldsymbol{x}=\boldsymbol{x}(t)}\right)\right\}\delta_{\zeta,\zeta'}\\&\quad+\frac{ec^2\hbar}{2\varepsilon^2}\left\{\langle\zeta,\boldsymbol{E}\times\boldsymbol{\beta}\rangle\frac{\boldsymbol{\beta}}{1+\gamma^{-1}}-\gamma\boldsymbol{\eta}\times\boldsymbol{E}-\boldsymbol{\eta}\times\boldsymbol{\beta}\frac{\langle\boldsymbol{\beta},\boldsymbol{E}\rangle}{1+\gamma^{-1}}\right.\\&\quad\left.+\boldsymbol{\beta}\left[\gamma^{-1}\boldsymbol{\eta}\boldsymbol{H}+\frac{\langle\boldsymbol{\beta},\boldsymbol{\eta}\rangle\langle\boldsymbol{\beta},\boldsymbol{H}\rangle}{1+\gamma^{-1}}\right]\right\}\\&\quad-\mathrm{i}\sqrt{\hbar}\langle\zeta',H_\nu|(\lambda_{pp}\hat{\mathscr{P}}_1\hat{G}-G^{+}\lambda_{pp}\hat{\mathscr{P}}_1)|H_\nu,\zeta\rangle+O(\hbar^2).\end{aligned}\tag{3.3}$$

Here

$$\sqrt{\hbar}\hat{G}=\sqrt{\hbar}\hat{\pi}_1+\langle\Delta\hat{x},d^2\hat{\mathscr{A}}\rangle(e/c\hbar3!)$$

$$\langle\chi(t)|\hat{G}^{+}|\varphi(t)\rangle=\langle\varphi(t)|\hat{G}|\chi(t)\rangle^{*}$$

and the last summands in formulas (3.2) and (3.3) can be written in the following form:

$$-\mathrm{i}\sqrt{\hbar}\langle\zeta', H_\nu|\left[\begin{pmatrix}\lambda_{pp}\hat{\mathscr{P}}_1\\ \Delta x\end{pmatrix}\hat{G}-\hat{G}^+\begin{pmatrix}\lambda_{pp}\hat{\mathscr{P}}_1\\ \Delta x\end{pmatrix}\right]|H_\nu,\zeta\rangle$$

$$=\frac{\hbar}{2}\sum_{k=1}^{3}\frac{1}{\operatorname{Im} b_k}\operatorname{Im}\left\{\begin{pmatrix}z_k\\ z_k\end{pmatrix}\int_0^t \mathrm{d}\tau\left\{\frac{e}{\varepsilon}\left\langle\boldsymbol{\eta},\left[\boldsymbol{E}(\tau)-\boldsymbol{\beta}\frac{\langle\boldsymbol{\beta},\boldsymbol{E}(\tau)\rangle}{1+\gamma^{-1}}\right]\right.\right.\right.$$

$$\left.\times\left[\boldsymbol{\beta}\frac{\gamma\langle\boldsymbol{\beta},\dot{z}_k^*\rangle}{1+\gamma^{-1}}+\dot{z}_k\right]\right\rangle$$

$$-\frac{ec}{2\gamma\varepsilon}\langle z_k^*\nabla\rangle\left\langle\boldsymbol{\eta},\left(\boldsymbol{\beta}\frac{\gamma\langle\boldsymbol{\beta},\boldsymbol{H}(x,\tau)\rangle}{1+\gamma^{-1}}+\boldsymbol{H}(x,\tau)\right)\right\rangle\bigg|_{x=x(\tau)}$$

$$\left.\left.+\frac{e\gamma}{\varepsilon}\langle\boldsymbol{\beta}\dot{z}_k^*\rangle\left\langle\boldsymbol{\eta},\left(\boldsymbol{\beta}\frac{\gamma\langle\boldsymbol{\beta},\boldsymbol{H}(\tau)\rangle}{1+\gamma^{-1}}+\boldsymbol{H}(\tau)\right)\right\rangle\right\}\right\}$$

$$+\frac{\hbar}{4}\delta_{\zeta,\zeta'}\sum_{k,j=1}^{3}\frac{2\nu_j+1}{\operatorname{Im} b_k\operatorname{Im} b_j}\operatorname{Im}\left\{\begin{pmatrix}\dot{z}_k\\ z_k\end{pmatrix}\{F(z_k^*, z_j^*, z_j)+F(z_k^*, z_j, z_j^*)\right.$$

$$\left.+F(z_j^*, z_k^*, z_j)+F(z_j, z_k^*, z_j^*)+F(z_j^*, z_j, z_k^*)+F(z_j, z_j^*, z_k^*)\}\right\},$$

where

$$3!\,F(z_j, z_k, z_m)=-\frac{1}{2}\int_0^t \mathrm{d}\tau\left\{\frac{e}{3!}\langle z_j,\nabla\rangle\langle z_k,\nabla\rangle\cdot\langle z_m,(\boldsymbol{E}(x,\tau)+\boldsymbol{\beta}(\tau)\times\boldsymbol{H}(x,\tau))\rangle\right.$$

$$-\frac{e}{c3!}[\langle z_j,\nabla)\cdot\langle(z_k\times\boldsymbol{H}(x,\tau)),\dot{z}_m\rangle+\langle z_m,\nabla\rangle\cdot\langle\dot{z}_j,(z_k\times\boldsymbol{H}(x,\tau))\rangle]$$

$$+\frac{\varepsilon\gamma^2}{2\varepsilon^2}[\langle\boldsymbol{\beta},\dot{z}_j\rangle(\langle\dot{z}_k,\dot{z}_m\rangle+\gamma^2\langle\boldsymbol{\beta},\dot{z}_k\rangle\langle\boldsymbol{\beta},\dot{z}_m\rangle)$$

$$\left.+(\langle\dot{z}_j,\dot{z}_k\rangle+\gamma^2\langle\boldsymbol{\beta},\dot{z}_j\rangle\langle\boldsymbol{\beta},\dot{z}_k\rangle)\langle\boldsymbol{\beta},\dot{z}_m\rangle]\right\}\bigg|_{x=x(\tau)}.$$

We denote by σ_{xx}, σ_{px} in formula (3.1) the 3×3-matrices of coordinate dispersions and coordinate and momentum correlations calculated with respect to quasi-classical TCS (1.3), respectively

$$\sigma_{xx}=\frac{\hbar}{4}[C(t)D_\nu C^+(t)+C^*(t)D_\nu C^t(t)]+O(\hbar^2),$$

$$\sigma_{px}=\frac{\hbar}{4}[B(t)D_\nu C^+(t)+B^*(t)D_\nu C^t(t)]+O(\hbar^2) \tag{3.4}$$

$$D_\nu=\left\|\frac{2\nu_k+1}{\operatorname{Im} b_k}\delta_{kj}\right\|_{3\times3}\qquad \sigma_{AB}=\frac{1}{2}\langle(\hat{A}\hat{B}+\hat{B}\hat{A})\rangle-\langle\hat{A}\rangle\langle\hat{B}\rangle.$$

The spectral-angle distribution of energy and the radiation probabilities can be obtained by common methods of quantum electrodynamics [1, 3]

$$\frac{d\varepsilon_{\rm rad}^{(\lambda)}}{d\Omega}=\frac{e^2}{4\pi^2 c}\int_0^\infty \omega^2\,d\omega \sum_{|\nu'|=0}^{\infty}(F_\lambda^{\uparrow\uparrow}\delta_{\zeta\zeta'}+F_\lambda^{\uparrow\downarrow}\delta_{\zeta-\zeta'}) \tag{3.5}$$

$$\frac{dw_{\rm rad}^{(\lambda)}}{d\Omega}=\frac{e^2}{4\pi^2\hbar c}\int_0^\infty \omega\,d\omega \sum_{|\nu'|=0}^{\infty}(F_\lambda^{\uparrow\uparrow}\delta_{\zeta\zeta'}+F_\lambda^{\uparrow\downarrow}\delta_{\zeta-\zeta'})$$

$$F_\lambda^{\uparrow\uparrow}=\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\langle e_\lambda, M^{\uparrow\uparrow}(t_1)\rangle\langle e_\lambda, M^{\uparrow\uparrow}(t_2)\rangle^*$$

$$F_\lambda^{\uparrow\downarrow}=\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\langle e_\lambda, M^{\uparrow\downarrow}(t_1)\rangle\langle e_\lambda, M^{\uparrow\downarrow}(t_2)\rangle^*. \tag{3.6}$$

Here the expansion in vectors e_λ, $\lambda=1,2$:

$$\begin{aligned} e_{(1)}&=e_\pi=(\cos\varphi\cos\theta,\ \sin\varphi\cos\theta,\ -\sin\theta)\\ e_{(2)}&=e_\sigma=(\sin\varphi,\ -\cos\varphi,\ 0)\end{aligned} \tag{3.7}$$

characterizes the polarization properties of radiation, and the matrix elements are given in the form of a sum of parts with spin flip and without spin flip

$$M(t,\zeta,\zeta',\hbar)=(\delta_{\zeta,\zeta'}+\delta_{\zeta,-\zeta'})M(t,\zeta,\zeta',\hbar)=\delta_{\zeta,\zeta'}M^{\uparrow\uparrow}+\delta_{\zeta,-\zeta'}M^{\uparrow\downarrow}$$

where $\delta_{\zeta,\zeta'}$ is the Kronecker symbol.

Let us sum the square of a matrix element over the finite states of the electron ν_1', ν_2', ν_3', then we obtain

$$\sum_{|\nu'|=0}^{\infty}F_\lambda^{\uparrow\uparrow}=\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\exp\left\{i\omega(t_1-t_2)-i\frac{\omega}{c}\langle nx(t_1,\hbar)\rangle+i\frac{\omega}{c}\langle nx(t_2,\hbar)\rangle\right\}$$
$$\times\left\{\frac{1}{c^2}\langle e_\lambda,\dot{x}(t_2,\hbar)\rangle\langle e_\lambda,\dot{x}(t_1,\hbar)\rangle+\hbar B_R^{(\lambda)}+\hbar B_\Phi^{(\lambda)}+\hbar B_S^{(\lambda)}\right\}+O(\hbar^2) \tag{3.8}$$

where

$$\begin{aligned}B_R^{(\lambda)}=\frac{i}{2c^2}\,[&\mathrm{Im}(\langle e_\lambda,\dot{C}(t_1)D_0\dot{C}^+(t_2)e_\lambda\rangle)+i\omega\langle e_\lambda,\beta(t_2)\rangle\,\mathrm{Im}(\langle e_\lambda,\dot{C}(t_1)D_0C^+(t_2)n\rangle)\\ &-i\omega\langle e_\lambda,\beta(t_1)\rangle\,\mathrm{Im}(\langle n,C(t_1)D_0\dot{C}^+(t_2)e_\lambda\rangle)\\ &+\omega^2\langle e_\lambda,\beta(t_1)\rangle\langle e_\lambda,\beta(t_2)\rangle\,\mathrm{Im}(\langle n,C(t_1)D_0C^+(t_2)n\rangle)]\delta_{\zeta,\zeta'}\end{aligned}$$

$$\begin{aligned}B_\Phi^{(\lambda)}=\frac{1}{2}\Bigg\{&\sum_{k,j=1}^{2}(-1)^{k+j+1}\langle e_\lambda,\beta(t_1)\rangle\langle\beta(t_2),e_\lambda\rangle\frac{\omega^2}{c^2}\,\mathrm{Re}(\langle n,C(t_k)D_\nu C^+(t_j)n\rangle)\\ &+\frac{(-i\omega)}{c^2}\sum_{k=1}^{2}(-1)^{k+1}[\langle e_\lambda,\beta(t_1)\rangle\,\mathrm{Re}(\langle n,C(t_1)D_\nu\dot{C}^+(t_k)e_\lambda\rangle)\\ &-\langle e_\lambda,\beta(t_2)\rangle\,\mathrm{Re}(\langle e_\lambda,\dot{C}(t_k)D_\nu C^+(t_2)n\rangle)]\\ &+\frac{1}{c^2}\,\mathrm{Re}(\langle e_\lambda,\dot{C}(t_1)D_\nu\dot{C}^+(t_2)e_\lambda\rangle)\Bigg\}\delta_{\zeta,\zeta'}\end{aligned}$$

$$B_S^{(\lambda)}=\mathrm{i}\,\frac{\omega}{2\varepsilon(t_1)}\,\langle e_\lambda,\boldsymbol{\beta}(t_2)\rangle\left[\langle\boldsymbol{\eta}\times\boldsymbol{\beta},e_\lambda\rangle\,\frac{\langle\boldsymbol{n},\boldsymbol{\beta}\rangle}{1+\gamma^{-1}}-\langle e_\lambda,\boldsymbol{\eta}\times\boldsymbol{n}\rangle+\langle\boldsymbol{\eta},\boldsymbol{n}\times\boldsymbol{\beta}\rangle\,\frac{\langle e_\lambda,\boldsymbol{\beta}\rangle}{1+\gamma^{-1}}\right](t_1)$$

$$-\mathrm{i}\,\frac{\omega}{2\varepsilon(t_2)}\,\langle e_\lambda,\boldsymbol{\beta}(t_1)\rangle\left[\langle e_\lambda,\boldsymbol{\eta}\times\boldsymbol{\beta}\rangle\,\frac{\langle\boldsymbol{n},\boldsymbol{\beta}\rangle}{1+\gamma^{-1}}\right.$$

$$\left.-\langle e_\lambda,\boldsymbol{\eta}\times\boldsymbol{n}\rangle+\langle\boldsymbol{\eta},\boldsymbol{n}\times\boldsymbol{\beta}\rangle\,\frac{\langle e_\lambda,\boldsymbol{\beta}\rangle}{1+\gamma^{-1}}\right](t_2)$$

$$\boldsymbol{x}(t,\hbar)=\boldsymbol{x}(t,\hbar,\zeta,\zeta). \tag{3.9}$$

For the spin flip, we obtain by similar calculations

$$\sum_{|\nu'|=0}^{\infty}F_\lambda^{\Downarrow}=\left|\int_{-\infty}^{\infty}\mathrm{d}t\,\exp\{\mathrm{i}[\omega t-(\omega/c)\langle\boldsymbol{n}\boldsymbol{x}(t)\rangle]\}\right.$$

$$\times\left\{\frac{1}{c}\,\langle e_\lambda,\dot{\boldsymbol{x}}(t,\zeta,-\zeta,\hbar)\rangle+\left(-\frac{\mathrm{i}\omega}{c}\right)\langle e_\lambda,\boldsymbol{\beta}\rangle\langle\boldsymbol{n},\boldsymbol{x}(t,\zeta,-\zeta,\hbar)\rangle\right.$$

$$+\mathrm{i}\,\frac{\hbar\omega}{2\varepsilon}\left\langle\boldsymbol{\eta}(t,\zeta,-\zeta)\right.$$

$$\left.\left.\left.\left[\boldsymbol{n}\times e_\lambda-\boldsymbol{n}\times\boldsymbol{\beta}\,\frac{\langle e_\lambda,\boldsymbol{\beta}\rangle}{1+\gamma^{-1}}+e_\lambda\times\boldsymbol{\beta}\,\frac{\langle\boldsymbol{n},\boldsymbol{\beta}\rangle}{1+\gamma^{-1}}\right]\right\rangle\right\}\right|^2+O(\hbar^3). \tag{3.10}$$

Thus, by taking into account the first quantum corrections, we can represent the radiated energy (3.5) in the form of a functional on a classical trajectory (see Introduction).

Let us consider briefly the characteristics of the quantum corrections in formula (3.8). The summand $B_S^{(\lambda)}$ describes the influence of spin on the radiation and is defined only by the classical trajectory, i.e. by the solutions of the Lorentz and Bargmann–Michel–Telegdi equations. As is shown below, the summand $B_\Phi^{(\lambda)}$ is also defined by the classical trajectory, more exactly, by the solutions of the Lorentz equations only, meanwhile the summand $B_\Phi^{(\lambda)}$ is related to the 'quantum character of the trajectory', namely, to quantum fluctuations of the basic variables $\boldsymbol{x}$ and $\hat{\boldsymbol{p}}$. More exactly, $B_\Phi^{(\lambda)}$ depends explicitly on the parameters of the initial state of a quantum particle, i.e. on the parameters of the wavepacket $\Psi_{\nu,\zeta}(\boldsymbol{x},t,\hbar)$ (1.3) localized as $\hbar\to 0$ in the neighbourhood of a classical trajectory; on the number ν which defines the wavepacket oscillations; and on the real and imaginary parts of the complex parameters $b_j, j=1,2,3$, which define, by (3.4), the width of the packet and the deviations of coordinates and momenta from their equilibrium states.

As shown in section 4, this summand can be neglected in the ultra-relativistic case, as can the 'fluctuation' part of the mean values $\boldsymbol{x}(t,\zeta,\zeta',\hbar)$, $\dot{\boldsymbol{x}}(t,\zeta,\zeta',\hbar)$ of the operator coordinates and velocities in the exponent and the first summand in formula (3.8).

The expressions (3.8)–(3.10) give, in principle, for an arbitrary field, the solution to the problem taking into account all the quantum corrections of the first order in $\hbar\to 0$ uniformly with respect to the relativism, and they allow one to consider the process of radiational self-polarization of electrons.

Remark. The existence of two types of quantum corrections in the electron radiation reflects the 'two-scale' character of quasi-classical asymptotics of wavefunctions (1.3) which were used in calculations of the radiational characteristics. When a classical Hamiltonian system is quantized by the method of the complex germ [37, 38, 42], two small dimensionless parameters appear (formally proportional to $\hbar$ and $\sqrt{\hbar}$, $\hbar\to 0$). The first of them, i.e. the ratio of the de Broglie wavelength to the characteristic dimensions of the system, defines rapid oscillations (with frequency $1/\hbar$) of the wavefunction, the second one characterizes the quantum fluctuations of a particle near a classical trajectory with frequency $\approx 1/\sqrt{\hbar}$. It was shown in [45] that expression (3.5) summed over the spin coincides with the corresponding expression for the spectral-angle distribution of the local power of radiation of a spinless relativistic particle.

4. Ultra-relativistic approximation

Consider the energy radiated (3.5) and the probabilities of transition with spin flip (3.6) in the ultra-relativistic approximation [4, 5], when the parameter $\gamma=(1-\beta^2)^{-1/2}\to\infty$. We restrict consideration to the case when the radiation in a given direction is formed by a part of the trajectory of length Δl, $\Delta l\approx O(\gamma^{-1})$ (for example, synchrotron radiation [2, 5]). In this case, as is known, essentially high frequencies are radiated

$$\omega=O(\gamma^3). \tag{4.1}$$

In the expression for spectral-angle distribution of the radiated energy (3.5) we introduce the new variables

$$t_1=t+\tau/2 \qquad t_2=t-\tau/2 \tag{4.2}$$

and the expression under the integral with respect to t will be considered as a spectral-angle distribution of the radiation power, $\mathrm{d}W^{(\lambda)}_{\mathrm{rad}}/\mathrm{d}\Omega$, taking into account that the main contribution to the integral is given by the domain of values small in τ [5]

$$\tau=O_\gamma(\gamma^{-1}) \qquad \gamma\to\infty. \tag{4.3}$$

By using estimates (4.1), (4.3) and expressions (1.18) we expand expressions (3.1), (3.3), (3.8) into a series in $\gamma^{-1}\to 0$. For the radiation power, after routine calculations [41] we finally have

$$\frac{dW^{(\lambda)}_{\mathrm{rad}}}{\mathrm{d}\Omega}=\frac{e^2}{4\pi^2 c}\int_0^\infty \omega^2\,\mathrm{d}\omega\int_{-\infty}^{\infty}\mathrm{d}\tau\,\langle \boldsymbol{e}_\lambda,\boldsymbol{\beta}(t_1)\rangle\langle \boldsymbol{e}_\lambda,\boldsymbol{\beta}(t_2)\rangle$$
$$\times\frac{\varepsilon}{\varepsilon-\hbar\omega}\exp\left[\frac{\mathrm{i}\omega\varepsilon}{\varepsilon-\hbar\omega}\tau\left(1-\langle \boldsymbol{n}\times\boldsymbol{\beta}\rangle+\frac{\tau^2\dot{\boldsymbol{\beta}}^2}{24}\right)\right]+O(\hbar^2). \tag{4.4}$$

This expression coincides (with accuracy to $O(\hbar^2)$) with the expression for spectral-angle distribution of the radiation power $\mathrm{d}W^{(\lambda)}_{\mathrm{rad}}/\mathrm{d}\Omega$ from [5].

Now we calculate the probability of transition with spin flip. By using estimates (4.1) and (4.3), we obtain

$$\eta(t_{1,2},\zeta,-\zeta)=\eta(t,\zeta,-\zeta)+O_\gamma(\gamma^{-1}) \qquad \langle \boldsymbol{e}_\lambda,\boldsymbol{\beta}\rangle=O_\gamma(\gamma^{-1})$$

and

$$\langle \boldsymbol{e}_\lambda,\boldsymbol{\beta}(t)\rangle\langle \boldsymbol{n},\boldsymbol{x}(t,\zeta,-\zeta)\rangle=O_\gamma(\gamma^{-2})$$
$$\boldsymbol{n}\times\boldsymbol{\beta}=O_\gamma(\gamma^{-1}) \qquad \langle \boldsymbol{n},\boldsymbol{\beta}\rangle=1+O_\gamma(\gamma^{-2}).$$

Then (3.10) can be reduced to the form

$$\sum_{|\nu'|=0}^{\infty} F_\lambda^{\uparrow\downarrow} = \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}\tau \, \exp\left[\mathrm{i}\omega\tau\left(1 - \langle \boldsymbol{n\beta}\rangle + \frac{\tau^2}{24}\dot{\boldsymbol{\beta}}^2\right)\right]$$
$$\times \frac{\hbar^2\omega^2}{4\varepsilon^2} \langle \boldsymbol{\eta}(t, \zeta, -\zeta), \boldsymbol{e}_\lambda \times \boldsymbol{q}_1\rangle \langle \boldsymbol{\eta}(t, \zeta, -\zeta), \boldsymbol{e}_\lambda \times \boldsymbol{q}_2\rangle + O_\lambda(\gamma^3) \tag{4.5}$$

where $\boldsymbol{q} = (1-\gamma^{-1})^{-1}\boldsymbol{\beta} - \boldsymbol{n}$. By using the relations

$$\boldsymbol{\eta}^*(t, \zeta, \zeta') = \boldsymbol{\eta}(t, \zeta', \zeta)$$
$$\langle \boldsymbol{a}, \boldsymbol{\eta}(t, \zeta, -\zeta)\rangle \langle \boldsymbol{b}, \boldsymbol{\eta}(t, \zeta, -\zeta)\rangle = \langle \boldsymbol{a}, \boldsymbol{b}\rangle - \langle \boldsymbol{\eta a}\rangle\langle \boldsymbol{\eta b}\rangle + \mathrm{i}\langle \boldsymbol{\eta}, \boldsymbol{a}\times\boldsymbol{b}\rangle$$
$$\boldsymbol{\eta} = \boldsymbol{\eta}(t, \zeta, \zeta)$$

and summing them over the states of polarization with the help of the identity

$$\sum_{\lambda=1}^{2} e_\lambda^j \cdot e_\lambda^k = \delta_{jk} - n_j \cdot n_k$$

for the probability of transition with spin flip, we obtain

$$\frac{\mathrm{d}w_{\mathrm{rad}}}{\mathrm{d}\Omega} = \frac{e_0^2\hbar}{4\pi^2} \int_0^{\infty} \omega^3 \, \mathrm{d}\omega \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}\tau \, \frac{1}{4\varepsilon^2(t)} \{\langle \boldsymbol{q}_1, \boldsymbol{q}_2\rangle (1 - \langle \boldsymbol{\eta}, \boldsymbol{n}\rangle^2)$$
$$+ \langle \boldsymbol{n}, \boldsymbol{\eta}\rangle [\langle \boldsymbol{n}, \boldsymbol{q}_1\rangle \langle \boldsymbol{\eta}, \boldsymbol{q}_2\rangle - \langle \boldsymbol{n}, \boldsymbol{q}_2\rangle\langle \boldsymbol{\eta}, \boldsymbol{q}_1\rangle] - \mathrm{i}\langle [\boldsymbol{\eta} - \langle \boldsymbol{n}, \boldsymbol{\eta}\rangle \boldsymbol{n}] \boldsymbol{q}_1 \times \boldsymbol{q}_2\rangle\} \tag{4.6}$$
$$\boldsymbol{q}_{1,2} = \boldsymbol{q}(t \pm \tau/2).$$

Formula (4.6) (with accuracy $O(\hbar^2)$) coincides with formula (14.4) from [5].

5. Non-relativistic approximations

Now we consider another limit case, when $\gamma \to 1$, $|\boldsymbol{\beta}| \to 0$, and hence, we can use the formal expansion in powers of $1/c$ [1, 3]. Expand expressions (3.3)–(3.7) into a series in $c^{-1} \to 0$ up to the highest-degree (in c^{-1}) term of the expansion (the dipole approximation). Then from (3.8) we obtain

$$\sum_{|\nu'|=0}^{\infty} F_\lambda^{\uparrow\uparrow} = \frac{1}{c^2} \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \, \exp\{\mathrm{i}\omega(t_1 - t_2)\} \{\langle \boldsymbol{e}_\lambda, \dot{\boldsymbol{x}}(t_1, \hbar)\rangle \langle \boldsymbol{e}_\lambda, \dot{\boldsymbol{x}}(t_2, \hbar)\rangle$$
$$+ \frac{\hbar}{2} \operatorname{Re}[\langle \boldsymbol{e}_\lambda, \dot{C}(t_1) D_\nu \dot{C}^+(t_2) \boldsymbol{e}_\lambda\rangle] + \mathrm{i}\frac{\hbar}{2} \operatorname{Im}[\langle \boldsymbol{e}_\lambda, \dot{C}(t_1) D_0 \dot{C}^+(t_2) \boldsymbol{e}_\lambda\rangle]$$
$$+ O(\hbar^2) + O(c^{-3}) \tag{5.1}$$

$$\dot{x}(t,\hbar)=c\beta-\frac{e\hbar}{2(3!)^2}\sum_{k,j=1}^{3}\frac{2v_j+1}{\operatorname{Im}b_k\operatorname{Im}b_j}$$

$$\times\operatorname{Im}\left\{\dot{z}_k(t)\int_0^t \mathrm{d}\tau\left\{\operatorname{Re}[\langle z_j(\tau),\nabla\rangle\langle z_j^*(\tau),\nabla\rangle]\langle z_k^*(\tau),E(x,\tau)\rangle\right.\right.$$

$$\left.\left.+2\langle z_k^*(\tau),\nabla\rangle\cdot\operatorname{Re}[\langle z_j(\tau),\nabla\rangle\langle z_j^*(\tau),E(x,\tau)\rangle]\right\}\Big|_{x=x(\tau)}\right\}$$

$$+O(\hbar^2)+O_c(c^{-3}).$$

Similarly, for (3.10), we obtain

$$\sum_{|v'|=0}^{\infty}F_\lambda^{\uparrow\downarrow}=\left|\int_{-\infty}^{\infty}\mathrm{d}t\exp\{\mathrm{i}\omega t\}\left[-\mathrm{i}\frac{\hbar\omega}{2mc^2}\langle\eta(t,\zeta,-\zeta)\cdot n\times e_\lambda\rangle\right.\right.$$

$$\left.\left.+\frac{(-\mathrm{i}\hbar)e}{2mc^2}\cdot\langle e_\lambda,a(t,\zeta,-\zeta)\rangle+O(\hbar^2)+O_c(c^{-3})\right]\right|^2 \tag{5.2}$$

where $a(t,\zeta,-\zeta)$ denotes

$$a(t,\zeta,-\zeta)=\sum_{k=1}^{3}\int_0^t\frac{\mathrm{d}\tau}{\operatorname{Im}b_k}[\dot{z}_k(t)\cdot\langle z_k^*(\tau),\nabla\rangle\langle\eta(\tau,\zeta,-\zeta)H(x,\tau)\rangle$$

$$-\dot{z}_k^*(t)\langle z_k(\tau),\nabla\rangle\langle\eta(\tau,\zeta,-\zeta)H(x,\tau)\rangle]\Big|_{x=x(\tau)}. \tag{5.3}$$

To find the total radiation energy and the probability of radiation with spin flip in the non-relativistic approximation we integrate expressions (5.1) and (5.2) over angles, using the well known relations [3, 5]

$$\frac{1}{2\pi}\oint \mathrm{d}\Omega\sum_\lambda\langle e_\lambda,a\rangle\langle e_\lambda,b\rangle=\frac{4}{3}\langle a,b\rangle$$

$$\frac{1}{2\pi}\oint \mathrm{d}\Omega\sum_\lambda\langle e_\lambda,a\rangle\langle b,e_\lambda\rangle\langle c,n\rangle=0$$

$$\frac{1}{2\pi}\oint \mathrm{d}\Omega\sum_{\lambda=1}^{2}\langle e_\lambda,a\rangle\langle e_\lambda,b\rangle\langle n,c\rangle\langle n,d\rangle$$

$$=\frac{2}{15}(4\langle a,b\rangle\langle c,d\rangle-\langle a,c\rangle\langle b,d\rangle-\langle a,d\rangle\langle b,c\rangle).$$

This, (3.5), and the relation

$$\int_0^\infty \mathrm{d}\omega\exp\{\mathrm{i}\omega\xi\}=\pi\delta(\xi)+P\frac{\mathrm{i}}{\xi}$$

give the following expression for the total radiated energy

$$\varepsilon_{\rm rad}=\frac{2}{3}\frac{e^2}{c^3}\left\{\int_{-\infty}^{\infty}\left[\langle\ddot{x}(t,\hbar),\ddot{x}(t,\hbar)\rangle+\frac{\hbar}{2}\,\mathrm{Re}(\mathrm{Sp}\;\ddot{C}(t)D_\nu\ddot{C}^+(t))\right]\mathrm{d}t\right.$$

$$\left.+\frac{\hbar}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}t_1\int_{-\infty}^{\infty}\mathrm{d}t_2\,\frac{1}{t_1-t_2}\,\mathrm{Im}(\mathrm{Sp}\;\ddot{C}(t_1)D_0\ddot{C}^+(t_2))\right\}$$

$$+O(\hbar^2)+O_c(c^{-4}). \tag{5.4}$$

Here

$$\ddot{x}(t,\hbar)=\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle x\rangle=\frac{\mathrm{d}}{\mathrm{d}t}\,\dot{x}(t,\hbar)$$

and $\dot{x}(t,\hbar)$ was defined in the dipole approximation in (5.1).

Similarly, using (3.6), we obtain the probability of radiation with spin flip

$$w_{\rm rad}^{\uparrow\downarrow}=\frac{e^2\hbar}{6\pi m^2c^5}\int_0^{\infty}\omega\,\mathrm{d}\omega\int_{-\infty}^{\infty}\mathrm{d}t_1\int_{-\infty}^{\infty}\mathrm{d}t_2\exp\{\mathrm{i}\omega(t_1-t_2)\}$$

$$\times\{\omega^2\langle\boldsymbol{\eta}(t_1,\zeta,-\zeta)\boldsymbol{\eta}^*(t_2,\zeta,-\zeta)\rangle+\langle\boldsymbol{\alpha}(t_1,\zeta,-\zeta)\boldsymbol{\alpha}^*(t_2,\zeta,-\zeta)\rangle\}$$

$$+O(\hbar^2)+O_c(c^{-6}). \tag{5.5}$$

or

$$w_{\rm rad}^{\uparrow\downarrow}=\frac{e^2\hbar}{6\pi m^2c^5}\int_0^{\infty}\omega\,\mathrm{d}\omega\;\{\omega^2|\boldsymbol{\eta}(\omega,\zeta,-\zeta)|^2+|\boldsymbol{\alpha}(\omega,\zeta,-\zeta)|^2\}+O(\hbar^2)+O_c(c^{-6})$$

where

$$\boldsymbol{\alpha}(\omega,\zeta,-\zeta)=\int_{-\infty}^{\infty}\mathrm{d}t\exp\{\mathrm{i}\omega t\}\boldsymbol{\alpha}(t,\zeta,-\zeta)$$

$$\boldsymbol{\eta}(\omega,\zeta,-\zeta)=\int_{-\infty}^{\infty}\mathrm{d}t\exp\{\mathrm{i}\omega t\}\boldsymbol{\eta}(\omega,\zeta,-\zeta)$$

are the Fourier images of the functions $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\eta}(t)$.

6. Examples

Consider a number of examples illustrating the results obtained. We restrict ourselves to expression (5.5) describing the spin effects of electron radiation using the non-relativistic approximation, since in the non-relativistic approximation the expression describing the total radiated energy of the electron (5.4) coincides with the similar result for the spinless particle obtained in [45]. By formula (5.4) in [45] and by taking into account the first quantum correction, the radiation power of a charged particle was calculated in a constant and homogeneous magnetic field, in the field of a harmonic oscillator and in [48] in an arbitrary focusing axially symmetric magnetic field. In particular, it was shown in [45] that for an appropriate choice of the initial trajectory-coherent state (namely, when this state is a correlated coherent state [14] or a coherent state [43, 44]) the expressions obtained for the power of spontaneous radiation in a constant and homogeneous magnetic field and the field of a harmonic oscillator coincide with the corresponding results of [14, 15].

6.1. Probability of radiation with spin flip (with non-relativistic approximation) for an electron moving in a constant and homogeneous magnetic field.

The equation of spin motion (1.16) in a homogeneous magnetic field $\boldsymbol{H}=(0, 0, H)$ using the non-relativistic approximation ($\varepsilon = mc^2$) is

$$\dot{\boldsymbol{\eta}} = \frac{e}{mc}\, \boldsymbol{\eta} \times \boldsymbol{H}$$

with the initial condition (1.16) for $\zeta' = -\zeta$ can be easily integrated

$$\boldsymbol{\eta}(t) = (C_1\, e^{i\omega_0 t} + C_2\, e^{-i\omega_0 t},\ iC_1\, e^{i\omega_0 t} - iC_2\, e^{-i\omega_0 t},\ C_3)^t \qquad \omega_0 = \frac{eH}{mc}$$
$$C_1 = \tfrac{1}{2}\, e^{-i\varphi}(\cos\theta + \zeta) \qquad C_2 = \tfrac{1}{2}\, e^{i\varphi}(\cos\theta - \zeta) \qquad C_3 = \sin\theta. \tag{6.1}$$

By substituting (6.1) into (5.5), we obtain

$$w_{\mathrm{rad}}^{\uparrow\downarrow} = \frac{e^2\hbar}{6\pi m^2 c^5} \int_0^\infty \omega^3 I(\omega, \zeta, -\zeta)\, d\omega$$
$$I(\omega, \zeta, -\zeta) = \int^\infty dt_1 \int^\infty dt_2\, \exp\{i\omega(t_1 - t_2)\} \langle \boldsymbol{\eta}(t_1, \zeta, -\zeta) \boldsymbol{\eta}^*(t_2, \zeta, -\zeta) \rangle \tag{6.2}$$
$$= 8\pi^2 |C_1|^2 \delta^2(\omega + \omega_0) + 4\pi^2 |C_3|^2 \delta^2(\omega) + 8\pi^2 |C_2|^2 \delta^2(\omega - \omega_0).$$

By passing in a standard way to the probability of radiation with spin flip in unit time $\bar{w}_{\mathrm{rad}}^{\uparrow\downarrow}$ we obtain

$$\bar{w}_{\mathrm{rad}}^{\uparrow\downarrow} = \lim_{T\to\infty} w_{\mathrm{rad}}^{\uparrow\downarrow}/T = \frac{e^2\hbar}{6\pi m^2 c^5}\, 4\pi\omega_0^2 |C_2|^2.$$

By using the explicit form of C_2, we obtain

$$\bar{w}_{\mathrm{rad}}^{\uparrow\downarrow} = \frac{\hbar e_0^2 \omega_0^2}{6 m^2 c^5} (\cos\theta - \zeta)^2 \qquad \zeta = \pm 1 \tag{6.3}$$

where θ already defines the electron spin orientation along the direction of the vector $\boldsymbol{l} = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$. We multiply (6.3) by the photon energy $\hbar\omega_0$ and sum over the states of polarization of the initial spin of the electron ($\zeta = \pm 1$). As a result, we obtain the total power of radiation of an electron with spin flip (in a homogeneous magnetic field)

$$p_{\mathrm{rad}}^{\uparrow\downarrow} = \frac{1}{6} \frac{\hbar^2 e^2 \omega_0^2}{m^2 c^5} (1 + \cos^2\theta). \tag{6.4}$$

For $\theta = 0$, this implies the result of [15].

6.2. Probability of radiation with spin flip (in non-relativistic approximation) for electron moving in an arbitrary focusing axially symmetric field

Consider the electron moving in an electric field with potential

$$A(x,t)=0 \qquad \Phi(x,t)=\Phi(r) \qquad r=(x^2+y^2)^{1/2} \tag{6.5}$$

then the classical trajectory is defined by the Hamiltonian system

$$\dot{x}=\mathscr{H}_p \qquad \dot{p}=-\mathscr{H}_x \qquad \mathscr{H}(p,x)=\frac{p^2}{2m}+e\Phi(r). \tag{6.6}$$

The system (6.6) admits families of circles as solution

$$\begin{aligned} &x(t)=R(\cos\omega_0 t, \sin\omega_0 t, 0) \qquad p(t)=m\dot{x}(t)\\ &\omega_0=\left(\frac{e}{mR}\frac{\partial\Phi(R)}{\partial R}\right)^{1/2} \end{aligned} \tag{6.7}$$

and R is defined by the condition $\mathscr{H}(p(t),x(t))=E$, where E is the energy of electron. The equation of spin motion for a given trajectory of electron has the form

$$\dot{\eta}(t)=-\frac{e}{2mc^2}\,\eta\times(\dot{x}(t)\times E(t)) \tag{6.8}$$

where the electric field calculated on the trajectory (6.7) has the form

$$E(t)=-\frac{\partial\Phi(R)}{\partial R}(\cos\omega_0 t, \sin\omega_0 t, 0).$$

System (6.8) with the initial condition (1.16) can be integrated for $\zeta'=-\zeta$ precisely as in the previous example:

$$\eta(t)=(C_1\,e^{i\Omega t}+C_2\,e^{-i\Omega t}, iC_1\,e^{i\Omega t}-iC_2\,e^{-i\Omega t}, C_3)^t$$

where $C_1=\frac{1}{2}e^{i\varphi}(\zeta-\cos\theta)$, $C_2=-\frac{1}{2}e^{-i\varphi}(\zeta+\cos\theta)$, $C_3=\sin\theta$, and the precession frequency of spin of the electron is

$$\Omega=\frac{e\omega_0}{mc^2}\frac{\partial\Phi(R)}{\partial R}.$$

Precisely as in the previous example, we obtain for the probability of transitions with spin flip in unit time

$$\bar{w}_{\rm rad}^{\uparrow\downarrow}=\frac{\hbar e^2|\Omega|^2}{6m^2c^5}(\zeta+\cos\theta)^2.$$

The effect of radiational self-polarization for relativistic electron–positron bundles in an axially symmetric electric field was considered in [46, 47].

7. Conclusions

By using quasi-classical trajectory-coherent states for the Dirac equation, we have constructed expressions for the characteristics of spontaneous radiation of an electron, with the first quantum correction included as a certain (absolutely specific) functional of a classical trajectory of a particle. Therewith, we understand the classical electron trajectory as a set of solutions of Lorentz and classical spin Bargmann–Michel–Telegdi equations.

The expression obtained for the first quantum correction holds for all energies of an electron (for which the quantum corrections are less than the corresponding classical term) and taken account of both the quantum loss of radiation and the fluctuational terms characterizing the quantum character of a trajectory. The part of the quantum correction which disappears in the ultra-relativistic approximation, and which from the very beginning was neglected in the operator method [4, 5], can be obtained here in explicit form. For non-relativistic particles this quantum correction is essential.

The dependence of the spin on the probability of radiation was obtained in the general form. In the limiting case of ultra-relativistic electrons, the result naturally coincides with that given in [5]. In the non-relativistic case, a comparatively simple general expression for the characteristics of spontaneous radiation of a charge was obtained with the spin properties of this charge taken into account.

We note that for ultra-relativistic particles the advantage of the quasi-classical operator method [4, 5] is that this method does not require that the quantum corrections for the loss of the photon be small; the use of quasi-classical trajectory–coherent states assumes that these quantum corrections are small. However, in our method there are no restrictions on the lowest particle energy limit. Thus the areas of application of these methods possess interesting intervals of values of physical parameters for the problem of spontaneous radiation of a charge.

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